International Journal of Theoretical Physics, Vol. 16, No. 11 (1977), pp. 845–861

Geodesics in Rotating Systems

D. G. ASHWORTH and P. A. DAVIES

Electronics Laboratories, University of Kent, Canterbury, Kent, England

Received: 29 *June* 1977

Abstract

The paper investigates the equations for geodesics, null geodesics, and spatial geodesics in rotating systems. Geodesics and null geodesics have, as usual, been interpreted as the paths of free particles and of light rays, respectively. Spatial geodesics are given a firm interpretation as the shortest paths between points within the rotating system, where the path length is measured by an observer in the rotating system who moves along the spatial geodesic. The paper shows that equations for geodesics in rotating systems may be derived by the traditional method, i.e., from the flat-space metric of general relativity, or by means of the instantaneous Lorentz frames approach. This supports the use of instantaneous Lorentz frames as a valid method for the analysis of events in rotating systems.

1. *Introduction*

In this paper we consider the derivation of equations that describe geodesics in rotating systems. These "geodesics" have been subdivided into three types: geodesics, null geodesics, and spatial geodesics. These three terms are interpreted as the path of a test particle, the path of a ray of light, and the shortest possible distance between any two points, respectively. The above definitions for geodesics and null geodesics are common in relativity theory and are accepted as fundamental assumptions. However, the identification of a spatial geodesic as the shortest possible distance between two points requires additional explanation. In a rotating system, as indeed in any system, there are various ways of measuring the distance between two points; one can use radar measurement, triangulation measurement, contiguous measurement using short measuring rods, and a variety of other techniques. However, if one has to actually move from one point to another there is only one path between the two points that will be shorter than any other, when measured by the observer who actually makes the journey. It is in this context that we shall discuss spatial geodesics.

The two main techniques for the analysis of rotating systems are the metric technique, as is common in general relativity, and the technique of instantaneous Lorentz frames. Ashworth and Jennison (1976) have already produced a restricted discussion of the applications of instantaneous Lorentz frames to rotating systems. It is one of the objectives of the present paper to show that the two seemingly unrelated approaches produce identical results.

In our discussion of rotating systems we introduce two systems of coordinates. One of these systems uses the frame S_2, which is related to the laboratory frame, S, by the Galilean rotational transformation, and the other system uses the frame S_1, which differs from S_2 only in angular measure. We show that S_1 and S_2 are equally valid for describing a rotating system and that the only difference between the two frames of reference is in the interpretation of measurements that have been made within the rotating system.

2. *The Metric Approach*

In this section we consider the derivation of the geodesic equations from the particular metric obtained when the Galilean rotational transformation is applied to the flat space-time of the Minkowski metric. The resulting metric corresponds, in particular, to a description of the rotating system made by an observer at the center of rotation of the system who is in synchronous rotation with the system. A restricted discussion of the interpretation, through the metric, of observations made by an observer who is rotating with the system has been made by Davies (1976).

The cylindrical form of the Minkowski metric in the laboratory system $S(r, \theta, z, t)$ is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 - c^2 dt^2 \tag{2.1}$$

and the Galilean rotational transformation to the frame $S_2(r_2, \theta_2, z_2, t_2)$ is given by

$$\begin{aligned} r &= r_2 \\ \theta &= \theta_2 + \omega t_2 \\ z &= z_2 \\ t &= t_2 \end{aligned} \tag{2.2}$$

whence we obtain in the frame S_2 the metric

$$ds^2 = dr_2^2 + r_2^2 d\theta_2^2 + dz_2^2 + 2\omega r_2^2 d\theta_2 dt_2 - (c^2 - \omega^2 r_2^2) dt_2^2 \tag{2.3}$$

The equations applicable to both geodesics and null geodesics are obtained from equation (2.3) in the usual manner and are of the form

$$\frac{d}{d\lambda}\{2\dot{r}_2\} = 2r_2\dot{\theta}_2^2 + 4\omega r_2\dot{\theta}_2\dot{t}_2 + 2\omega^2 r_2 \dot{t}_2^2 \tag{2.4}$$

$$\frac{d}{d\lambda}\{2r_2^2\dot{\theta}_2 + 2\omega r_2^2 \dot{t}_2\} = 0 \tag{2.5}$$

$$\frac{d}{d\lambda}\{2\dot{z}_2\} = 0 \tag{2.6}$$

and

$$\frac{d}{d\lambda}\{2\omega r_2^2\dot{\theta}_2 - 2(c^2 - \omega^2 r_2^2)\dot{t}_2\} = 0 = \frac{d}{d\lambda}\left\{\frac{\partial}{\partial \dot{t}_2}(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)\right\}$$
$$= \frac{d}{d\lambda}\left\{\frac{\partial}{\partial \dot{t}_2}\left(\frac{ds}{d\lambda}\right)^2\right\} \tag{2.7}$$

in which λ is a parameter that varies along the curve and the dots denote differentiation with respect to λ.

Integrating equations (2.5)-(2.7) and denoting the constants of integration by $2A$, $2B$, and $2D$, respectively, gives

$$\dot{t}_2 = (A\omega - D)/c^2 \tag{2.8}$$

$$\dot{\theta}_2 = (Ac^2 - A\omega^2 r_2^2 + \omega r_2^2 D)/r_2^2 c^2 \tag{2.9}$$

and

$$\dot{z}_2 = B \tag{2.10}$$

Equation (2.3) can also be written in the form

$$\left(\frac{ds}{d\lambda}\right)^2 = \dot{r}_2^2 + r_2^2\dot{\theta}_2^2 + \dot{z}_2^2 + 2\omega r_2^2\dot{\theta}_2\dot{t}_2 - (c^2 - \omega^2 r_2^2)\dot{t}_2^2 \tag{2.11}$$

Let us now confine our attention to the plane $z_2 = 0$.

2.1 *Geodesics.* The geodesics of the metric given in equation (2.3) are fully described by equations (2.8)-(2.11) together with the condition $ds/d\lambda = 1$ thus giving

$$\frac{d\theta_2}{dr_2} = \left[\frac{a_2}{r_2(r_2^2 - a_2^2)^{1/2}} \pm \frac{\omega r_2}{u(r_2^2 - a_2^2)^{1/2}}\right] \tag{2.12}$$

in which u is the velocity of a particle as measured in the laboratory frame $S(r, \theta, z, t)$ such that

$$u = \frac{(1 + c^2\dot{t}_2^2)^{1/2}}{\dot{t}_2} \tag{2.13}$$

Also, $r_2 = a_2$ at the point of the closest approach of the geodesic to the origin, i.e., at the point where $dr_2/d\theta_2 = 0$.

If u is independent of r, then equation (2.12) may be integrated to give

$$\theta_2 = \pm\cos^{-1}(a_2/r_2) \pm (\omega/u)(r_2^2 - a_2^2)^{1/2} + K_1 \tag{2.14}$$

in which K_1 is a constant of integration, such that $K_1 = \theta_2$ when $r_2 = a_2$, and in which any combination of the signs is permissible.

Equation (2.14) is the first equation in Table 1 and is the equation of a geodesic, i.e., the path of a free particle, in the frame S_2, as calculated from the metric of equation (2.3), assuming that the particle traveled through the laboratory frame with a constant velocity u.

TABLE 1. A summary of the equations describing the loci of geodesics, null geodesics, and spatial geodesics in the rotating systems S_1 and S_2. The methods whereby the equations were derived are also indicated

Property	Frame	Equation	Possible methods of derivation
Geodesic	S_2	$\theta_2 = \pm[\cos^{-1}(a_2/r_2) - (\omega/u)(r_2^2 - a_2^2)^{1/2}]$	(1) From the path of a particle traveling at velocity u in the laboratory frame S (2) From the metric (3) By transforming the differential equation for the geodesic in S_1 into the frame S_2
Null geodesic	S_2	$\theta_2 = \pm[\cos^{-1}(a_2/r_2) - (\omega/c)(r_2^2 - a_2^2)^{1/2}]$	(1) From the path of a particle traveling at velocity c in the laboratory frame S (2) From the metric (3) By transforming the differential equation for the null geodesic in S_1 into the frame S_2
Spatial geodesic	S_2	$\theta_2 = \pm[\cos^{-1}(a_2/r_2) - (a_2\omega^2/c^2)(r_2^2 - a_2^2)^{1/2}]$	(1) From the definition of a spatial geodesic as the *shortest possible* distance between two points (2) From the metric (3) By transforming the differential equation for the spatial geodesic in S_1 into the frame S_2

Geodesic	S_1	$\theta_1 = \pm\left\{\sin^{-1}\left[\dfrac{c(r_1^2 - a_1^2)^{1/2}}{r_1(c^2 - a_1^2\omega^2)^{1/2}}\right] - \dfrac{c}{u}\sin^{-1}\left[\dfrac{\omega(r_1^2 - a_1^2)^{1/2}}{(c^2 - a_1^2\omega^2)^{1/2}}\right]\right\}$	(1)	From the technique of instantaneous frames
			(2)	By transforming the differential equation for the geodesic in S_2 into the frame S_1
Null geodesic	S_1	$\theta_1 = \pm\cos^{-1}\left[\dfrac{a_1 c + \omega r_1^2}{r_1(c + a_1\omega)}\right]$	(1)	From the technique of instantaneous frames with a particle moving at the velocity of light
			(2)	By transforming the differential equation for the null geodesic in S_2 into the frame S_1
Spatial geodesic	S_1	$\theta_1 = \pm\left\{\cos^{-1}\left[\dfrac{a_1(1 - r_1^2\omega^2/c^2)^{1/2}}{r_1(1 - a_1^2\omega^2/c^2)^{1/2}}\right] - \dfrac{\lvert a_1\omega\rvert}{c} \times \cos^{-1}\left[\dfrac{(1 - r_1^2\omega^2/c^2)^{1/2}}{(1 - a_1^2\omega^2/c^2)^{1/2}}\right]\right\}$	(1)	From the shortest distance as measured by an observer who moves towards a point in the rotating system (i.e., using the technique of instantaneous frames)
			(2)	By transforming the differential equation for the spatial geodesic in S_2 into the frame S_1
			(3)	From the definition of a spatial geodesic as the *shortest possible* distance between two points

2.2. *Null Geodesics.* The equation of a null geodesic, found by solving equations (2.8)-(2.11) and setting $ds/d\lambda = 0$, is given by

$$\theta_2 = \pm\cos^{-1}(a_2/r_2) \pm (\omega/c)(r_2^2 - a_2^2)^{1/2} + K_1 \qquad (2.15)$$

where a_2 and K_1 have the same meaning as previously defined and, once again, any combination of the signs is permissible. Equation (2.15) is the second equation in Table 1.

Alternatively one can, of course, consider a light ray as being composed of photons traveling with velocity $u = c$, which, when substituted into equation (2.14), again produces equation (2.15).

2.3. *Spatial Geodesics.* If we define a spatial geodesic as the shortest distance between any two points in *space*, rather than in space-time, then we have the additional condition that

$$\frac{\partial s}{\partial t_2} = 0 \qquad (2.16)$$

Equation (2.16) may be written in the form $\partial\dot{s}/\partial\dot{t}_2 = 0$, which, when multiplied by $2\dot{s}$, gives the condition that

$$\frac{\partial}{\partial\dot{t}_2}\left\{\left(\frac{ds}{d\lambda}\right)^2\right\} = 0 \qquad (2.17)$$

which, by equation (2.7), means that $D = 0$. Equations (2.8) and (2.12) can then be solved to give $u = c^2/a\omega$, which, when combined with equation (2.14) gives

$$\theta_2 = \pm\ \{\cos^{-1}(a_2/r_2) - (a_2\omega^2/c^2)(r_2^2 - a_2^2)^{1/2}\} + K_1 \qquad (2.18)$$

for the equation of a spatial geodesic. K_1 is, as before, a constant of integration. This equation has previously been derived, but without a physical interpretation, by Arzeliès (1966) and is listed as the third equation in Table 1. As the velocity $u = c^2/a\omega$ of the particle in the laboratory system is greater than the velocity of light it is obvious that no real free particle can ever travel along a spatial geodesic in the frame S_2.

3. *The Instantaneous Lorentz Frames Approach*

Before any analysis is performed it is instructive to examine exactly what is meant by instantaneous frames of reference. For any body moving under an acceleration it is always possible to find a frame moving with constant velocity relative to, for example, the laboratory frame, in which the body is instantaneously at rest. If it is now assumed that the equations relating measurements in the accelerating and nonaccelerating frames are the same at any instant of time as those relating measurements in rectilinearly moving frames, then it is possible to analyze events in the accelerating system. We should note that the use of instantaneous frames automatically includes the clock hypothesis, i.e., the rate of a moving clock, in its own frame and as seen from any other frame, is not affected by any acceleration imposed upon it.

3.1. *Geodesics.* We adopt the definition of a geodesic in a rotating system as being the path of a test particle across the system and commence our investigations by interpreting this path according to observers who are rotating with the system. This interpretation of the path depends upon the way in which a rotating observer makes a measurement. We do not assume or derive, in this section, any "transformation to a rotating system" by means of which the equation of the path of a particle in the laboratory frame may be transformed to the rotating system. Instead, we use the method of associating each point in the rotating system, instantaneously, with a linearly moving Lorentz frame.

Consider a cylindrically symmetric system of an infinity of infinitely close reference frames. At any instant we can let any of these frames be coincident with linearly moving Lorentz frames and hence we may consider the infinity of reference frames as representing an infinity of instantaneous Lorentz frames. We now allow this system of reference frames to rotate about a point 0 with angular velocity ω as measured in the laboratory frame S. S has the usual cylindrical coordinates, r, θ, z, t. The system is constrained to rotate in the plane $z = 0$ with center of rotation, 0, at $r = 0$. We define the angular velocity, ω, by stating that if one of the instantaneous rotating frames is maintained at a distance r from 0, as measured in the laboratory frame S, then the instantaneous linear velocity, v, of the rotating frame is ωr, in S.

We shall also allow a system of cylindrical coordinates $S_1(r_1, \theta_1, z_1, t_1)$ to be synchronized to the rotating system such that $r_1 = 0$ is located at 0. Hence, all the rotating instantaneous frames will appear at rest in S_1. We shall assume that $\theta_1 = 0$ coincides with $\theta = 0$ at time $t_1 = t = 0$ and that all the rotating instantaneous frames are aligned such that at the instant of observation in S_1 their z axes all point in the same direction, their y axes are all in radial directions, and their x axes are all in tangential directions, as seen in S_1.

Consider now a particle that moves through this rotating system of Lorentz frames. Let us assume that the path of this particle is a straight line of length σ in the laboratory system, S, and that its velocity in the laboratory system is u. What is the equation of the path of the particle in the system S_1?

As there is an infinity of Lorentz frames in the rotating system the particle will pass through the origins of some of these frames. The angles between the path of the particle and the y axis of each of these frames may be calculated from the Lorentz transformations or, alternatively, from the aberration equations. Also, because these frames are at rest in the system S_1, these angles may be measured by observers in the rotating system and a locus of the origins of the frames through which the ray has passed may be produced for the coordinate system S_1.

Figure 1 shows the situation as seen from the laboratory frame, S.

Consider now a particle that passes through the origins of the instantaneous frames S_{1n} and $S_{1(n+1)}$. The path of the particle, as seen in the frame S_1, is depicted in Figure 2. From this figure it is evident that

$$\tan \phi_{1n} = \frac{r_1 d\theta_1}{dr_1}, \qquad \cos \phi_{1n} = \frac{dr_1}{d\sigma_1} \tag{3.1}$$

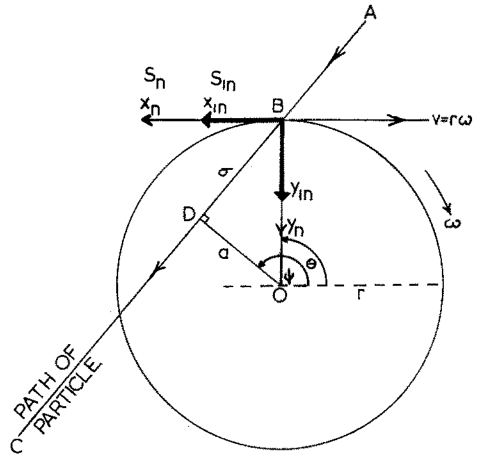

Figure 1. The path, ABC, of a particle as seen in the laboratory frame S. S_n is a Lorentz frame instantaneously at rest in S, and S_{1n} is a Lorentz frame instantaneously at rest in the rotating system S_1. The origins of S_n and S_{1n} are momentarily coincident at B as the particle passes through B. The section of path BD is of length σ.

If the particle starts in S at time $t = t_1 = 0$ and at an angle $\theta = \theta_1 = 0$, then, for the nth instantaneous frame, S_{1n}, in S_1 through which the particle passes, it is always possible to associate a frame S_n, which is at rest in the laboratory frame S, in standard configuration with S_{1n}, and with origins coincident at $t_{1n} = t_n = 0$ such that the equation of the path of the particle in S_n is (as may

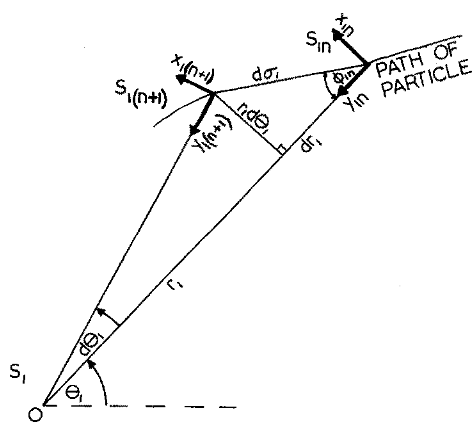

Figure 2. The path of the particle as seen in the frame S_1 when the particle has passed through the origins of S_{1n} and $S_{1(n+1)}$.

be seen from Figure 1) given by

$$x_n = \frac{aut_n}{r}, \qquad y_n = \pm\frac{u}{r}(r^2 - a^2)^{1/2}t_n \tag{3.2}$$

The particle passes through the origins of S_n and S_{1n} at $t_n = t_{1n} = 0$ and the sign of x_n and y_n depends upon the direction in which the particle is traveling. The path of the particle between the origins of the two frames S_n and S_{n+1}, as measured in the laboratory frame S, will be a straight line of length $d\sigma$ and will subtend an angle $d\theta$ at the center of rotation of the system. From equation (3.2), $d\sigma = udt$ where dt is the time taken for the particle to travel from the origin of S_n to the origin of S_{n+1}.

ϕ_{1n} may now be calculated by using the Lorentz transformations (Einstein, 1905):

$$x_{1n} = \frac{x_n - vt_n}{(1 - v^2/c^2)^{1/2}}, \qquad y_{1n} = y_n, \qquad z_{1n} = z_n \tag{3.3}$$

from which

$$\tan\phi_{1n} = \frac{dx_{1n}}{dy_{1n}} = \pm\frac{c(au - rv)}{u(r^2 - a^2)^{1/2}(c^2 - v^2)^{1/2}} \tag{3.4}$$

Let us now consider two of the instantaneous rotating frames, S_{1a} and S_{1b}, located on the same radius in S_1 at radial distances r_{1a} and r_{1b}. It is possible to associate frames S_a and S_b at radial distances r_a and r_b in the frame S instantaneously with the frames S_{1a} and S_{1b}. It is also possible to measure the distance between S_{1a} and S_{1b} in S_1 or the distance between S_a and S_b in S. As the relative motion of S_a to S_{a1} or S_{b1}, and the relative motion of S_b to S_{a1} or S_{b1}, is at right angles to the direction of r_{1a}, r_{1b}, r_a, and r_b, observers in each of the frames will agree that $r_{1a} = r_a$ and $r_{1b} = r_b$, which may be generalized to give

$$r_1 = r \tag{3.5}$$

Combining equations (3.1) and (3.4) and remembering that $v = r\omega$ gives

$$d\theta_1 = \pm\frac{c(a_1u - \omega r_1{}^2)\,dr_1}{ur_1(r_1{}^2 - a_1{}^2)^{1/2}(c^2 - r_1{}^2\omega^2)^{1/2}},$$

$$d\sigma_1 = \frac{cr_1\,[(u - a_1\omega)^2 + \omega^2(r_1{}^2 - a_1{}^2)(1 - u^2/c^2)]^{1/2}dr_1}{u(r_1{}^2 - a_1{}^2)^{1/2}(c^2 - r_1{}^2\omega^2)^{1/2}} \tag{3.6}$$

in which u will, in general, be a function of r. In the special case that u is a constant the first of equations (3.6) may be integrated directly to yield the fourth equation of Table 1.

Let us also examine two of the instantaneous rotating frames, S_{1A} and S_{1B}, located at equal radial distances from 0 but on different radii, in S_1. If the angle $d\theta_1$ between these two radii is very small then S_{1A} and S_{1B} become virtually the same frame since their velocities relative to the laboratory frame become equal in magnitude and direction. If an instantaneous Lorentz frame in S, at a distance $r = r_1$ from 0, is associated with the frames S_{1A} and S_{1B}, then, since $dx = rd\theta$ and $dx_1 = r_1 d\theta_1$, equation (3.3) gives

$$d\theta_1 = \frac{d\theta - \omega dt}{(1 - v^2/c^2)^{1/2}} \tag{3.7}$$

where v is the relative velocity of S_{1A} or S_{1B} to S_A or S_B. It is interesting to compare the differential equations given in equation (3.6) above with those obtained using the Galilean rotational transformation. This transformation can be used to relate a frame $S_2(r_2, \theta_2, z_2, t_2)$ to the laboratory frame S by using the transformations of equation (2.2). The system S_2 corresponds to a coordinate system, with origin located at 0, which rotates with respect to the laboratory frame with an angular velocity ω, but has the same angular measure as the laboratory frame. It is the frame obtained by making a Galilean rotation of S about itself. Combining equations (2.2), (3.5), and (3.7) we find that

$$\begin{aligned} dr_1 &= dr_2 \\ d\theta_1 &= \frac{d\theta_2}{(1 - v^2/c^2)^{1/2}} \end{aligned} \tag{3.8}$$

Now, combining equations (3.6) and (3.8) and remembering that $v = r\omega$, we obtain

$$\begin{aligned} d\theta_2 &= \pm \frac{(a_2 u - \omega r_2^2) dr_2}{u r_2 (r_2^2 - a_2^2)^{1/2}}, \\ d\sigma_2 &= \frac{c r_2 [(u - a_2\omega)^2 + \omega^2 (r_2^2 - a_2^2)(1 - u^2/c^2)]^{1/2} dr_2}{u (r_2^2 - a_2^2)^{1/2} (c^2 - r_2^2 \omega^2)^{1/2}} \end{aligned} \tag{3.9}$$

where $d\sigma_2$ is an element of the path of the particle in S_2 and is defined by

$$(d\sigma_2)^2 = \frac{(r_2 d\theta_2)^2}{(1 - \omega^2 r_2^2/c^2)} + (dr_2)^2$$

and u will, in general, be a function of r. In the special case that u is a constant, the first of equations (3.9) may be integrated directly to yield the first equation of Table 1.

3.2. *Null Geodesics.* In this section a null geodesic in a rotating system is defined as the path of a light ray as it crosses the system. The path and its length are given in differential form, in S_1, by equation (3.6) after letting $u = c$, the velocity of light. Making this substitution and integrating between the limits $r_1 = r_1$ and $r_1 = a_1$ to obtain the locus, and length, of the path of the ray in the system S_1 we find that

$$\theta_1 = \pm \cos^{-1}\left\{\frac{a_1 c + \omega r_1^2}{r_1(c + a_1\omega)}\right\} \tag{3.10a}$$

and

$$\sigma_1 = \pm \frac{1}{\omega}(c - a_1\omega)\sin^{-1}\left\{\frac{(\omega/c)(r_1^2 - a_1^2)^{1/2}}{(1 - \omega^2 a_1^2/c^2)^{1/2}}\right\} \tag{3.10b}$$

in which the sign of ω may be either positive or negative and the sign before equation (3.10b) is chosen such that σ_1 is positive. Equation (3.10a), which is listed as the fifth equation in Table 1, may be seen to be the equation of the curve HBD in Figure 3 and σ_1 is the length of the circular arc BD. From Figure 3 it is evident that

$$\alpha_1 = \pm \cos^{-1}\left(\frac{m_1 \omega}{c \pm \omega a_1}\right) \tag{3.11a}$$

and

$$\sigma_1 = \left(\frac{c \pm a_1\omega}{\omega}\right)\sin^{-1}\left(\frac{m_1\omega}{c \pm a_1\omega}\right) \tag{3.11b}$$

which is identical to equation (8) of Ashworth and Jennison (1976) and represents the arc of a circle of radius $\frac{1}{2}[c/\omega \pm a_1]$ with center $\frac{1}{2}[c/\omega \mp a_1]$ from 0, in which the sign depends upon the sign of ω.

Equations (3.11a) and (3.11b) describe the ray path which a rotating observer would derive by measuring the aberration angle of the light as it passed through his or her own infinitely small locality. An infinite number of such contiguous measurements made at all points along the light path would give, when drawn by the observer on a piece of paper in his or her own locality, i.e., when drawn in the local Euclidean space, the light path as defined by equations (3.11a), (3.11b).

The path of the same ray of light in the frame S_2 is obtained by setting $u = c$ in equation (3.9), which may then be integrated between the limits $r_2 = r_2$ and $r_2 = a_2$ to obtain the locus, and length, of the path of the ray in the system S_2, thus giving

$$\theta_2 = \pm \cos^{-1}\left(\frac{a_2}{r_2}\right) \pm \frac{\omega}{c}(r_2^2 - a_2^2)^{1/2} \tag{3.12a}$$

and

$$\sigma_2 = \pm \frac{1}{\omega}(c - a_2\omega)\sin^{-1}\left\{\frac{(\omega/c)(r_2^2 - a_2^2)^{1/2}}{(1 - \omega^2 a_2^2/c^2)^{1/2}}\right\} \tag{3.12b}$$

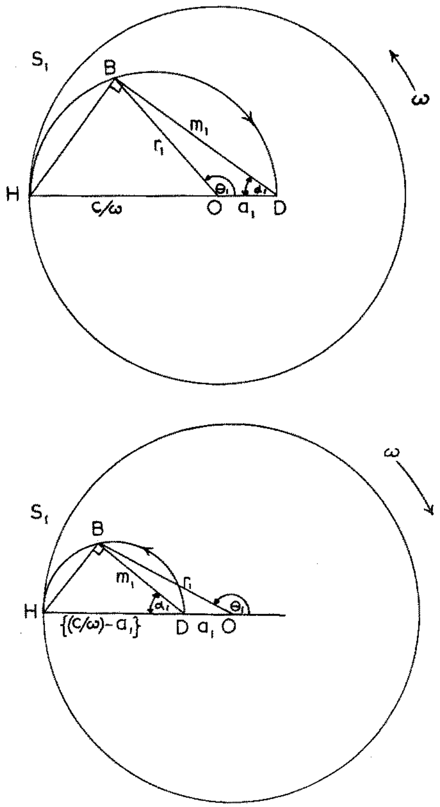

Figure 3. The geometry of a ray path in the rotating system S_1. The ray path follows the circular arc HBD in the upper diagram and the circular arc DBH in the lower diagram.

in which the sign of ω may be either positive or negative and the sign before equation (3.12b) is chosen such that σ_2 is positive.

Equation (3.12a) is precisely the same equation as may be obtained by transforming the equation of the path of a light ray in S directly to S_2 by the Galilean transformation of equation (2.2): From Figure 1, assuming a constant velocity u for the particle, we have

$$r\cos(\theta - \psi) = \pm a$$
$$r\sin(\theta - \psi) = \pm ut$$

which, together with equation (2.2), gives

$$\theta_2 = \pm\cos^{-1}\left(\frac{a_2}{r_2}\right) \pm \frac{\omega}{u}(r_2^2 - a_2^2)^{1/2} + \psi$$

which becomes identical to equation (3.12a) if we specify $u = c$ for a light ray and $\theta_2 = 0$ when $r_2 = a_2$. Equation (3.12a) has previously been derived by Arzeliès (1966) and is equivalent to equation (2.15), which we derived in Section 2.2.

It is therefore possible to describe the path of a light ray as the arc of a circle in the system S_1, or as an Archimedean spiral in system S_2, where S_1 and S_2 are related by equation (3.8). Both descriptions are equally "correct", and the choice of which system should be used is arbitrary although in practice the system into which experimental measurements could most easily be substituted would always be used. If measurements were being made by an observer as he or she crossed the rotating system, then S_1 would be used, but if measurements were made by an observer situated at the center of the rotating system then S_2 would be used.

3.3. *Spatial Geodesics.* We define a spatial geodesic as *the* shortest distance between two points in the rotating system. This is the shortest distance, measured piecemeal, according to an observer who moves between the two points in the rotating system. There can be only one such spatial geodesic linking any two points, and all spatial geodesics must be independent of the direction of rotation of the system.

In common with the metric approach given in Section 2.3 we shall consider a spatial geodesic to be defined as the path, in the rotating system, of a test particle that is traveling at a constant velocity in the laboratory frame. Since a spatial geodesic has a minimal path length in *space* it will correspond to the path traveled by a particle that is moving at infinite velocity as measured in the rotating system. Using the velocity addition formula it can readily be shown that the velocity, u_1, of a particle as measured in the system S_1 is related to the velocity, u, of the same particle measured in the laboratory system S by

$$u_1 = \frac{[(c^2 - \omega a_1 u)^2 - (c^2 - u^2)(c^2 - \omega^2 r_1{}^2)]^{1/2}}{c(1 - \omega a_1 u/c^2)} \tag{3.13}$$

u_1 will thus be a maximum, i.e., equal to infinity, when $u = c^2/a_1\omega$. The velocity in the laboratory system of the test particle whose paths determine the spatial geodesics of the rotating system must therefore be given by

$$u = c^2/a_{1,2}\,\omega \tag{3.14}$$

[It is worth noting that the equation of a spatial geodesic can also be obtained by letting

$$u = \omega r_{1,2}^2 c^2/a_{1,2}(2c^2 - \omega^2 r_{1,2}^2) \tag{3.15}$$

but in this case the spatial geodesic is not the path of a single particle traveling with constant velocity in the laboratory frame.]

The fact that equation (3.14) requires a particle velocity that is greater than the velocity of light does not invalidate the procedure, as no real particle is required to travel along a spatial geodesic path—it is simply the path of a hypothetical test particle. Letting $(d\sigma_1)_{u=c^2/a_1\omega} = dl_1$ enables equation (3.6) to be integrated between the limits $r_1 = r_1$ and $r_1 = a_1$ to give

$$\theta_1 = \pm\left\{\cos^{-1}\left[\frac{a_1(1 - r_1^2\omega^2/c^2)^{1/2}}{r_1(1 - a_1^2\omega^2/c^2)^{1/2}}\right] - \frac{|a_1\omega|}{c}\cos^{-1}\left[\frac{(1 - r_1^2\omega^2/c^2)^{1/2}}{(1 - a_1^2\omega^2/c^2)^{1/2}}\right]\right\} \tag{3.16a}$$

and

$$l_1 = (r_1^2 - a_1^2)^{1/2}(1 - a_1^2\omega^2/c^2)^{1/2} \tag{3.16b}$$

Equation (3.16a), which is the sixth equation listed in Table 1, is the equation of the locus of a spatial geodesic in S_1. Equation (3.16b), which has been obtained previously by Ashworth and Jennison (1976), gives the length of a spatial geodesic in S_1.

Letting $(d\sigma_2)_{u=c^2/a_2\omega} = dl_2$ enables equation (3.9) to be integrated between $r_2 = r_2$ and $r_2 = a_2$,

$$\theta_2 = \pm\left\{\cos^{-1}\left(\frac{a_2}{r_2}\right) - \frac{a_2\omega^2}{c^2}(r_2^2 - a_2^2)^{1/2}\right\} \tag{3.17a}$$

and

$$l_2 = (r_2^2 - a_2^2)^{1/2}(1 - a_2^2\omega^2/c^2)^{1/2} \tag{3.17b}$$

Equation (3.17a) is the equation of the locus of a spatial geodesic in S_2; it has previously been derived by Arzeliès (1966) and is equivalent to equation (2.18), which we derived in Section 2.3.

An alternative method of deriving equations (3.16) and (3.17) is to use the equations previously derived to describe the path of a light ray across a rotating system, together with simple geometrical considerations.

Figure 4 is a representation of the path HBD of the light ray in the frame S_1. If we let an observer at point B compute the shortest distance to D, accord-

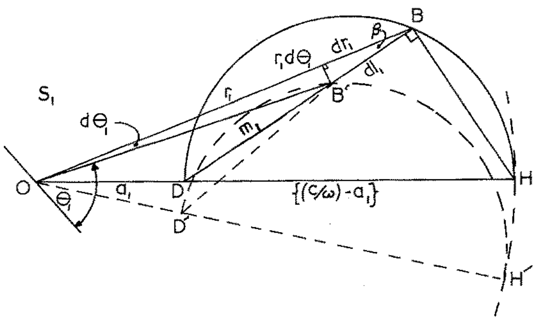

Figure 4. Construction used in Section 3.3 to describe an infinitesimal spatial geodesic, BB′.

ing to measurements made in the system S_1, then this shortest distance will be along the straight line BD. If this same observer now moves an infinitesimal distance BB′ along the line BD then a different light ray H′B′D′ will pass through his new position and the shortest distance to D′ lies along the straight line B′D′. Continuing this process the observer follows the path of a spatial geodesic which has a distance of closest approach to 0 of a_1 and the path may be computed from the geometry of Figure 4 as follows:

$$\tan\beta = r_1 \frac{d\theta_1}{dr_1}, \qquad \cos\beta = \frac{dr_1}{dl_1} \tag{3.18}$$

and

$$\cos\beta = \frac{m_1^2 + r_1^2 - a_1^2}{2r_1 m_1} \tag{3.19}$$

But, also from the geometry of Figure 4,

$$m_1 = \frac{(r_1^2 - a_1^2)^{1/2}(c - a_1\omega)^{1/2}}{(c + a_1\omega)^{1/2}}$$

giving, by equation (3.19),

$$\cos\beta = \frac{(r_1^2 - a_1^2)^{1/2} c}{(c^2 - a_1^2\omega^2)^{1/2} r_1}$$

Hence, from equation (3.18),

$$d\theta_1 = \frac{a_1(c^2 - r_1^2\omega^2)^{1/2} dr_1}{c r_1 (r_1^2 - a_1^2)^{1/2}}, \qquad dl_1 = \frac{r_1(c^2 - a_1^2\omega^2)^{1/2} dr_1}{c(r_1^2 - a_1^2)^{1/2}} \tag{3.20}$$

Equations (3.20) integrate to give equations (3.16) which describe the path of a spatial geodesic in S_1. The equations of the spatial geodesic in the system S_2 may be found by using equations (3.8) and (3.20) giving

$$d\theta_2 = \frac{a_2(c^2 - r_2^2\omega^2)\, dr_2}{r_2 c^2 (r_2^2 - a_2^2)^{1/2}}, \qquad dl_2 = \frac{r_2(c^2 - a_2^2\omega^2)^{1/2} dr_2}{c(r_2^2 - a_2^2)^{1/2}} \tag{3.21}$$

where dl_2 is an element of the spatial geodesic in S_2 defined by $(dl_2)^2 = (r_2 d\theta_2)^2/(1 - \omega^2 r_2^2/c^2) + (dr_2)^2$. Equations (3.21) integrate to give equations (3.17), which describe the path of a spatial geodesic in S_2.

We submit that, unlike Arzeliès, who used a purely mathematical approach towards spatial geodesics, the treatment we have presented above puts spatial geodesics on a firm physical basis by explaining them as explicit geometrical entities.

4. *Conclusions*

We have derived the equations describing geodesics, null geodesics, and spatial geodesics in the systems S_1 and S_2 of a rotating system. These derivations can first be made in either of the two systems and then transformed to the other system if this is desired. The resulting equations are summarized in Table 1. In order to simplify the table the equations for $\theta_{1,2} = f(r_{1,2})$, but not the equations for $\sigma_{1,2} = f(r_{1,2})$, are presented.

The work in the paper shows that the systems S_1 and S_2 are equally valid for describing events in a rotating system and also shows that there is a simple relationship between S_1 and S_2. It is important to realize, however, that S_2, which is obtained by a Galilean transformation from the laboratory frame S, is applicable to an observer at the center of rotation of the system who is spinning in synchronism with the system. The frame S_1 corresponds to the interpretation of events made by an observer who is in synchronous rotation with the system and who actually moves through it, making measurements as he or she goes. The question of whether to use S_1 or S_2 to analyze any particular problem in a rotating system is arbitrary but will, in a practical case, depend on the position of the observer within the rotating system. For example, consider the path of a ray of light that travels from an observer at the center of the system to an observer in synchronous rotation at the circumference of the system. Both observers will, of course, use their own local coordinate systems for making measurements. A Lorentz frame instantaneously at rest with the circumferential observer will provide a simple way of calculating, for example, the aberration angle of the incoming ray as seen by this observer. It would therefore be natural for the observer to use the system of instantaneous frames S_1 to describe events. However, the observer fixed at the center of the disk would tend to use S_2 since he or she would interpret the path of the ray as an Archimedean spiral. So, although the choice of S_1 or S_2 is entirely arbitrary for a theoretical analysis of any problem in a rotating system, in practice the choice is likely to depend upon the position of the observer.

We are not suggesting that the approach to measurements in rotating systems as used in this paper is necessarily better than the ideal general relativity approach where a metric applicable to an observer on a rotating disk is derived through Einstein's field equations. However, a universally agreed metric of this type does not, as yet, exist, and it is necessary to find some other method for investigating measurements in rotating systems. We submit that the method of instantaneous Lorentz frames provides a very reasonable method of interpreting such measurements and we hope that this paper has clarified the relationship between the commonly used frame S_2 and the frame S_1 which results when instantaneous Lorentz transformations are applied to rotating systems. In order to emphasize the fact that different methods of analysis produce the same answers, we have provided a table which clearly shows the different methods that may be used to derive identical equations.

References

Arzeliès, H. (1966). *Relativistic Kinematics*, Pergamon, Oxford, Chap. 9.
Ashworth, D. G., and Jennison, R. C. (1976). *Journal of Physics A*, **9**, 35.
Davies, P. A. (1976). *Journal of Physics A*, **9**, 951.
Einstein, A. (1905). *Annalen der Physik*, **17**, 891.